

ON JACOBI FIELD SPLITTING THEOREMS

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ABSTRACT. We formulate extensions of Wilking's Jacobi field splitting theorem to uniformly positive sectional curvature and also to positive and nonnegative intermediate Ricci curvatures.

In [Wilk1], Wilking established the following remarkable Jacobi field splitting theorem.

Theorem A. (*Wilking*) *Let γ be a unit speed geodesic in a complete Riemannian n -manifold M with nonnegative curvature. Let Λ be an $(n-1)$ -dimensional space of Jacobi fields orthogonal to γ on which the Riccati operator S is self-adjoint. Then Λ splits orthogonally into*

$$\Lambda = \text{span}\{J \in \Lambda \mid J(t) = 0 \text{ for some } t\} \oplus \{J \in \Lambda \mid J \text{ is parallel}\}.$$

This result has several impressive applications, so it is natural to ask about analogs for other curvature conditions. We provide these analogs for positive sectional curvature and also for nonnegative and positive intermediate Ricci curvatures. For positive curvature our result is the following.

Theorem B. *Let M be a complete n -dimensional Riemannian manifold with $\text{sec} \geq 1$. For $\alpha \in [0, \pi)$, let $\gamma : [\alpha, \pi] \rightarrow M$ be a unit speed geodesic, and let Λ be an $(n-1)$ -dimensional family of Jacobi fields orthogonal to γ on which the Riccati operator S is self-adjoint. If*

$$\max\{\text{eigenvalue } S(\alpha)\} \leq \cot \alpha,$$

then Λ splits orthogonally into

$$(1) \quad \text{span}\{J \in \Lambda \mid J(t) = 0 \text{ for some } t \in (\alpha, \pi)\} \oplus \{J \in \Lambda \mid J = \sin(t)E(t) \text{ with } E \text{ parallel}\}.$$

Notice that for $\alpha = 0$, the boundary inequality, $\max\{\text{eigenvalue } S(\alpha)\} \leq \cot \alpha = \infty$, is always satisfied. So Bonnet's theorem follows as a corollary.

To understand the initial value hypothesis, $\max\{\text{eigenvalue } S(\alpha)\} \leq \cot \alpha$, we note that in the model case when M is the unit n -sphere and $\Lambda = \text{span}\{J \mid J(0) = 0\}$, $S(t) = \cot(t) \cdot \text{id}$. So Theorem B says that if $\max\{\text{eigenvalue } S(\alpha)\}$ is smaller than in the model, then

$$\begin{aligned} \Lambda &= \text{span}\{J \in \Lambda \mid J(t) = 0 \text{ for some } t \in (\alpha, \pi]\} \\ &= \text{span}\{J \in \Lambda \mid J(t) = 0 \text{ for some } t \in (\alpha, \pi)\} \oplus \{J \in \Lambda \mid J = \sin(t)E(t) \text{ with } E \text{ parallel}\}. \end{aligned}$$

Like Theorem A, and in contrast to the Rauch Comparison Theorem, Theorem B has no hypothesis about conjugate points. Geodesics in Complex, Quaternionic, and Octonionic projective spaces with their canonical metrics provide explicit examples of Theorem B in the presence of conjugate points and also show that both summands in (1) can be nontrivial.

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Examples 1.6 and 1.7 (below) show that Theorem B is optimal in the sense that neither the boundary inequality nor the hypothesis that $S|_\Lambda$ is self-adjoint can be removed from the statement.

Recall ([Wu], [Shen]) that a Riemannian manifold M is said to have k^{th} -Ricci curvature $\geq \iota$ provided that for any choice $\{v, w_1, w_2, \dots, w_k\}$ of an orthonormal $(k+1)$ -frame, the sum of sectional curvatures $\sum_{i=1}^k \sec(v, w_i)$ is $\geq \iota$. In short hand, this is written as $\text{Ric}_k M \geq \iota$. Clearly $\text{Ric}_k M \geq \iota k$ implies $\text{Ric}_{k+1} M \geq \iota(k+1)$. $\text{Ric}_1 M \geq \iota$ is the same as $\sec M \geq \iota$, and $\text{Ric}_{n-1} M \geq \iota$ is the same as $\text{Ric} M \geq \iota$.

When the hypothesis $\sec_M \geq 0$ is replaced with $\text{Ric}_k(M) \geq 0$ and

$$\dim\{J \in \Lambda \mid J(t) = 0 \text{ for some } t\} \leq n - k - 1,$$

we get the following interpolation between Wilking's Jacobi field splitting theorem and Theorem 1.7.1 in [GromWal].

Theorem C. *Let γ be a unit speed geodesic in a complete Riemannian n -manifold M with $\text{Ric}_k(\dot{\gamma}) \geq 0$. Let Λ be an $(n-1)$ -dimensional space of Jacobi fields orthogonal to γ on which the Riccati operator S is self-adjoint. If*

$$(2) \quad \dim\{\text{span}\{J \in \Lambda \mid J(t) = 0 \text{ for some } t\}\} \leq n - k - 1,$$

then Λ splits orthogonally into

$$\Lambda = \text{span}\{J \in \Lambda \mid J(t) = 0 \text{ for some } t\} \oplus \{J \in \Lambda \mid J \text{ is parallel}\}.$$

In particular, if we also have $\text{Ric}_k(\dot{\gamma}) > 0$, then

$$\dim\{\text{span}\{J \in \Lambda \mid J(t) = 0 \text{ for some } t\}\} \geq n - k.$$

For sectional curvature, the Hypothesis (2) is

$$\dim\{\text{span}\{J \in \Lambda \mid J(t) = 0 \text{ for some } t\}\} \leq n - 2.$$

If this is not satisfied, we have $\dim\{\text{span}\{J \in \Lambda \mid J(t) = 0 \text{ for some } t\}\} = n - 1$, or

$$\Lambda = \text{span}\{J \in \Lambda \mid J(t) = 0 \text{ for some } t\}.$$

Thus Theorem C extends Theorem A to the Ric_k case.

In the Ricci curvature case, Hypothesis (2) says that all nonzero Jacobi fields along γ are nowhere vanishing, so for Ricci curvature, Theorem C becomes the following result from [GromWal].

Theorem D. *(Theorem 1.7.1 in [GromWal]) Let γ be a unit speed geodesic in a complete Riemannian n -manifold M with nonnegative Ricci curvature. Let Λ be an $(n-1)$ -dimensional space of Jacobi fields orthogonal to γ on which the Riccati operator S is self-adjoint. Let $L(t) \equiv \{J(t) \mid J \in \Lambda\}$. If $L(t)$ spans $\dot{\gamma}(t)^\perp$ for all $t \in \mathbb{R}$, then $S \equiv 0$, $\sec(\dot{\gamma}, \cdot) \equiv 0$, and Λ consists of parallel Jacobi fields.*

Just as Wilking applied Theorem A to obtain a structure result for metric foliations in positive curvature, we apply Theorem C to obtain a structure result for metric foliations of manifolds with positive Ric_k . First we recall the definitions of metric foliations and their dual leaves from [GromWal] and [Wilk1].

Definition E. A metric foliation \mathcal{F} of a Riemannian manifold M is a partition of M into connected subsets, called leaves, that are locally equidistant in the following sense. For all $p \in M$, there are neighborhoods $U \subset V$ of p so that for any two leaves L_1 and L_2 and connected components N_1 and N_2 of $L_1 \cap V$ and $L_2 \cap V$, respectively, the function

$$\begin{aligned} \text{dist}_{N_1} &: N_2 \cap U \longrightarrow \mathbb{R}, \\ \text{dist}_{N_1}(q) &\equiv \text{dist}(q, N_1) \end{aligned}$$

is constant.

Examples include the fiber decomposition of a Riemannian submersion and the orbit decomposition of an isometric group action.

A unit speed geodesic $\gamma : (0, \infty) \longrightarrow M$ is called horizontal for \mathcal{F} if and only if for all $t_0 > 0$, there is an $\varepsilon > 0$ so that

$$\text{dist}(L(\gamma(t_0)), \gamma(t)) = |t - t_0|$$

for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$. Here $L(\gamma(t_0))$ is the leaf containing $\gamma(t_0)$.

The *Dual Leaf through* p is defined to be

$$\mathfrak{L}^\#(p) \equiv \{q \in M \mid \text{there is a piece-wise smooth horizontal curve from } p \text{ to } q\}.$$

Theorem F. Let \mathcal{F} be a Riemannian foliation of a complete Riemannian n -manifold E with $\text{Ric}_k(E) > 0$. Then the dimension of the leaves of $\mathcal{F}^\#$ are all $\geq n - k + 1$.

Sectional curvature is the case when $k = 1$. So our conclusion is that the dimension of the leaves of $\mathcal{F}^\#$ are all $\geq n$. In other words, there is only one leaf as shown by Wilking in Theorem 1 of [Wilk1].

For the Ricci curvature case, our conclusion is that the dimension of the leaves of $\mathcal{F}^\#$ are all ≥ 2 . For an alternative proof of this fact, combine Theorem D with the argument on the top of page 1305 of [Wilk1].

Remark. Let \mathcal{F} be a Riemannian foliation of a complete nonnegatively curved manifold. Wilking showed that the dual foliation of \mathcal{F} is also Riemannian if its leaves are complete. One might speculate that the same holds for Riemannian foliations of manifolds with $\text{Ric}_k \geq 0$ if the leaves of $\mathcal{F}^\#$ are complete and their dimensions are all $\leq n - k$. Wilking's proof uses the Rauch Comparison Theorem in a crucial way and hence is not directly applicable to the Ric_k case.

For uniformly positive intermediate Ricci curvature we will prove the following analog of Theorem B.

Theorem G. Let M be an n -dimensional, complete Riemannian manifold with $\text{Ric}_k \geq k$. For $\alpha \in [0, \pi)$, let $\gamma : [\alpha, \pi] \longrightarrow M$ be a unit speed geodesic. Let Λ be an $(n - 1)$ -dimensional family of Jacobi fields on which the Riccati operator S is self-adjoint. If

$$\max\{\text{eigenvalue } S(\alpha)\} \leq \cot \alpha$$

and if

$$\dim\{J \in \Lambda \mid J(t) = 0 \text{ for some } t\} \leq n - k - 1,$$

then Λ splits orthogonally into

$$\text{span}\{J \in \Lambda \mid J(t) = 0 \text{ for some } t \in (\alpha, \pi)\} \oplus \{J \in \Lambda \mid J = \sin(t)E(t) \text{ with } E \text{ parallel}\}.$$

In contrast to Ricci curvature, the Ric_k condition has the disadvantage that it is not given by a tensor. On the other hand, constructing examples with positive Ric_k with existing techniques seems much easier than constructing examples with positive sectional curvature. For example, many more normal homogeneous spaces have $Ric_k > 0$ than have positive sectional curvature.

The proof of Theorem B begins with the following extension of D to positive Ricci curvature.

Theorem H. *Let γ be a unit speed geodesic in a complete Riemannian n -manifold M with $Ric \geq n-1$. For $\alpha \in [0, \pi)$, let Λ be an $(n-1)$ -dimensional family of Jacobi fields orthogonal to a unit speed geodesic $\gamma : [\alpha, \pi] \rightarrow M$ on which the associated Riccati operator S is self-adjoint. Let $L \equiv \{J(t) \mid J \in \Lambda\}$ and assume that L spans $\dot{\gamma}(t)^\perp$ for $t \in (\alpha, \pi)$. If*

$$\max\{\text{eigenvalue } S(\alpha)\} \leq \cot \alpha,$$

then $S \equiv \cot(t) \cdot \text{id}$ and consequently $\sec(\dot{\gamma}, \cdot) \equiv 1$.

Notice that for $\alpha = 0$, the boundary inequality, $\max\{\text{eigenvalue } S(\alpha)\} \leq \cot \alpha = \infty$, is always satisfied. So Myer's Theorem follows as a corollary. Just as we can view Theorem D as an infinitesimal version of the Splitting Theorem [CheegGrom], Theorem H can be viewed as an infinitesimal version of Cheng's maximal diameter theorem [Cheng].

Section 1 begins with the proofs of Theorems B and H and concludes with examples that show that Theorem B is optimal in the sense that neither the boundary inequality nor the hypothesis that $S|_\Lambda$ is self-adjoint can be removed from the statement. The proofs of Theorems C, F, and G are given in Section 2.

Remark. *See [VerdZil] for other interesting extensions of Wilking's Jacobi field results.*

It seems that Theorem C could be derived from Theorem B of [VerdZil], but since our proof of Theorem C is so simple, we have not studied its relationship to Theorem B of [VerdZil] in detail. It might also be possible to prove our Theorem B using the techniques of [VerdZil].

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1. UNIFORMLY POSITIVE CURVATURE

First we work on the proof of Theorem H. Set

$$\begin{aligned} s &\equiv \frac{1}{n-1} \text{trace}(S), \\ S_0 &\equiv S - \frac{\text{trace}(S)}{n-1} \cdot \text{id}, \\ r &\equiv \frac{1}{n-1} (Ric(\dot{\gamma}, \dot{\gamma}) + |S_0|^2), \text{ and} \end{aligned}$$

abusing notation, let

$$R(\cdot) = R(\cdot, \dot{\gamma}) \dot{\gamma}.$$

Recall ([GromWal], page 36) that the Jacobi equation is equivalent to the two first order equations

$$(1.0.3) \quad \begin{aligned} S(J) &= J' \text{ and} \\ S^2 + S' + R &= 0, \end{aligned}$$

and that Equation 1.0.3 implies

$$s^2 + s' + r = 0.$$

Proposition 1.1. *Assume the hypotheses of Theorem H. If $s(t) \equiv \cot(t)$, then $S(t) \equiv \cot(t) \cdot id$ and therefore $\sec(\gamma'(t), \cdot) \equiv 1$.*

Proof. Substituting $s(t) \equiv \cot(t)$ into $s^2 + s' + r = 0$ gives

$$\begin{aligned} 0 &= \cot^2(t) - \csc^2(t) + r \\ &= -1 + r. \end{aligned}$$

So $r \equiv 1$. Consequently

$$\begin{aligned} 1 &= r \\ &= \frac{Ric(\dot{\gamma}, \dot{\gamma}) + |S_0|^2}{n-1} \\ &\geq \frac{(n-1) + |S_0|^2}{n-1} \\ &= 1 + \frac{|S_0|^2}{n-1}. \end{aligned}$$

Thus $|S_0| \equiv 0$, and

$$\begin{aligned} S &= \frac{\text{trace}(S)}{n-1} \cdot id \\ &= s \cdot id \\ &= \cot(t) \cdot id. \end{aligned}$$

Substituting $S = \cot(t) \cdot id$ into the Riccati equation, $S^2 + S' + R = 0$, gives

$$\begin{aligned} (\cot^2(t) - \csc^2(t)) \cdot id + R &= 0, \\ -id + R &= 0, \end{aligned}$$

and therefore $\sec(\gamma', \cdot) \equiv 1$. □

Remark 1.2. *The solution to the initial value problem*

$$f^2 + f' + 1 = 0; \quad f(t_0) = s(t_0), \quad t_0 \in [0, \pi]$$

is

$$f(t) = \cot(t - (t_0 - \cot^{-1} s(t_0))).$$

If $s(t_0) < \cot(t_0)$, it follows that f has an asymptote at a time $t = d \in (t_0, \pi)$, and $\lim_{t \rightarrow d^-} f(t) = -\infty$. If $s(t_0) > \cot(t_0)$, it follows that f has an asymptote at a time $t = d \in (0, t_0)$, and $\lim_{t \rightarrow d^+} f(t) = \infty$.

Proposition 1.3. *Assume the hypotheses of Theorem H. Let f be as in Remark 1.2 for some $t_0 \in [\alpha, \pi]$. Then*

- (a) $s|_{[\alpha, t_0)} \geq f|_{[\alpha, t_0)}$ and
- (b) $s|_{(t_0, \pi)} \leq f|_{(t_0, \pi)}$.

Proof. Set $y = f - s$ on (α, t_0) . Using $f^2 + f' + 1 = 0$ and $s^2 + s' + r = 0$ we find

$$\begin{aligned} y' &= f' - s' \\ &= -f^2 + s^2 + r - 1 \\ &= -(f - s)(f + s) + r - 1. \end{aligned}$$

So y solves the initial value problem

$$(1.3.1) \quad y' = -(f + s)y + r - 1, \quad y(t_0) = 0.$$

Let x be a nontrivial solution to $x' = -\frac{1}{2}(f + s)x$ and u a function that satisfies $u' = \frac{2(r-1)}{x^2}$ and $u(t_0) = 0$. We claim that this implies $y = \frac{1}{2}ux^2$. To justify this claim, we'll show $\frac{1}{2}ux^2$ solves the initial value problem 1.3.1. Indeed,

$$\begin{aligned} \left(\frac{1}{2}ux^2\right)' &= \frac{1}{2}(u'x^2 + 2uxx') \\ &= \frac{1}{2}\left(\frac{2(r-1)}{x^2}x^2 - ux(f + s)x\right) \\ &= (r - 1) - \frac{1}{2}ux^2(f + s). \end{aligned}$$

Since $u(t_0) = 0$, our claim holds and $y = \frac{1}{2}ux^2$.

Since $r \geq 1$, $u' = \frac{2(r-1)}{x^2} \geq 0$. Combined with $u(t_0) = 0$, we see that $u \leq 0$ on $[\alpha, t_0)$. Hence $y|_{[\alpha, t_0)} = \frac{1}{2}ux^2|_{[\alpha, t_0)} \leq 0$, and $s|_{[\alpha, t_0)} \geq f|_{[\alpha, t_0)}$, proving Part a.

Similarly, $u' \geq 0$ and $u(t_0) = 0$ yield $u|_{(t_0, \pi)} \geq 0$. Hence $s|_{(t_0, \pi)} \leq f|_{(t_0, \pi)}$ as claimed. \square

We are now ready to prove Theorem H.

Proof of Theorem H. Assume, for contradiction, that $S \not\equiv \cot(t) \cdot id$. It follows from Proposition 1.1 that $s(t) \not\equiv \cot(t)$.

Assume for the moment that $s(t_0) < \cot(t_0)$ for some $t_0 \in [\alpha, \pi]$. Then by Remark 1.2, f has an asymptote at a time $d \in (t_0, \pi)$, and $\lim_{t \rightarrow d^-} f(t) = -\infty$. On the other hand, by Proposition 1.3(b), $s|_{(t_0, \pi)} \leq f|_{(t_0, \pi)}$. So s is not defined for all $t \in (\alpha, \pi)$. This contradicts our hypothesis that $L \equiv \{J(t) | J \in \Lambda\}$ spans $\dot{\gamma}(t)^\perp$ for $t \in (\alpha, \pi)$.

We may assume therefore $s(t) \geq \cot(t)$ on $[\alpha, \pi)$. Since we assumed

$$\max\{\text{eigenvalue } S(\alpha)\} \leq \cot \alpha,$$

it follows that $s(\alpha) = \cot(\alpha)$. So either, $s \equiv \cot$ or for some $t_0 \in (\alpha, \pi]$, $s(t_0) > \cot(t_0)$. Assume the latter. Then by Remark 1.2, the graph of f has an asymptote on $(0, t_0)$. If the asymptote for f lies in $(0, \alpha)$, then the graph of f is a shift to the right of the graph of \cot by an amount in $(0, \alpha)$. Combining this with Proposition 1.3(a) gives

$$s(\alpha) \geq f(\alpha) > \cot(\alpha) = s(\alpha),$$

a contradiction.

The remaining case is when $s(t_0) > \cot(t_0)$ and the asymptote for f is in (α, t_0) . By Proposition 1.3(a),

$$s|_{[\alpha, t_0)} \geq f|_{[\alpha, t_0)},$$

and by Remark 1.2,

$$\lim_{t \rightarrow d+} f(t) = \infty.$$

Together the previous two displays imply that s is not defined at d , contrary to our hypothesis that $L \equiv \{J(t) | J \in \Lambda\}$ spans $\dot{\gamma}(t)^\perp$ for $t \in (\alpha, \pi)$. Hence $s \equiv \cot$, and by Proposition 1.1, $S \equiv \cot(t) \cdot id$ and $\sec(\gamma'(t), \cdot) \equiv 1$ as desired. \square

We complete the proof of Theorem B by mostly following the lines of Wilking's proof of Theorem A. In particular, we use Wilking's generalization of the Horizontal Curvature Equation from [Wilk1], which we review in outline below. For details see [GromWal].

Let γ be a unit speed geodesic in a complete Riemannian n -manifold M . Let Λ be an $(n-1)$ -dimensional space of Jacobi fields orthogonal to γ on which the Riccati operator S is self-adjoint. Let Ψ be any vector subspace of Λ . Define

$$V(t) \equiv \{J(t) \mid J \in \Psi\} \oplus \{J'(t) \mid J \in \Psi, J(t) = 0\}.$$

Note that the second summand vanishes for almost every t , and $V(t)$ defines a smooth distribution along γ ([GromWal], Lemma 1.7.1, [Wilk1], page 1300). Set

$$H(t) \equiv V(t)^\perp \cap \dot{\gamma}(t)^\perp.$$

At a time t_0 that satisfies $V(t_0) = \{J(t_0) \mid J \in \Psi\}$, we define a Riccati operator

$$\begin{aligned} \hat{S} &: H(t_0) \longrightarrow H(t_0) \text{ by} \\ \hat{S}(y) &= \left((J^h)' \right)^h|_{t_0}, \text{ where } J \in \Lambda \text{ satisfies } J(t_0) = y, \end{aligned}$$

and the superscript h denotes the component in H . We also define a map

$$\begin{aligned} A &: V(t_0) \longrightarrow H(t_0) \text{ by} \\ A(u) &= (J')^h(t_0), \text{ where } J \in \Psi, J(t_0) = u. \end{aligned}$$

Theorem 1.4. (Wilking) \hat{S} and A are well-defined maps for all $t \in \mathbb{R}$, \hat{S} is self-adjoint, and \hat{S} and A satisfy the Riccati type equation

$$(1.4.1) \quad \hat{S}' + \hat{S}^2 + \{R(\cdot, \dot{\gamma}(t)) \dot{\gamma}(t)\}^h + 3AA^* = 0.$$

Remark 1.5. To see how this generalizes the Horizontal Curvature Equation of [Gray] and [O'Neill], suppose γ is a horizontal geodesic for a Riemannian submersion $\pi : M \rightarrow B$. For Λ take the Jacobi fields that correspond to variations of horizontal geodesics that leave $\pi^{-1}(\pi(\gamma(0)))$ orthogonally. For Ψ take the Holonomy fields, that is those that come from the lifts of γ . Then via the Horizontal Curvature Equation, Equation 1.4.1 becomes the usual Riccati Equation along $\pi(\gamma)$.

Proof of Theorem B. Let M be an n -dimensional Riemannian manifold with $\sec \geq 1$. For $\alpha \in [0, \pi)$, let $\gamma : [\alpha, \pi] \rightarrow M$ be a unit speed geodesic. Let Λ be an $(n-1)$ -dimensional family of Jacobi fields on which the Riccati operator S is self-adjoint and satisfies

$$\max\{\text{eigenvalue } S(\alpha)\} \leq \cot \alpha.$$

Set

$$\Psi \equiv \{J \in \Lambda \mid J(t) = 0 \text{ for some } t\},$$

and define $V(t)$, $H(t)$, the Riccati operator $\hat{S} : H(t) \rightarrow H(t)$ and the map $A : V(t) \rightarrow H(t)$ as above.

By Theorem 1.4.1, \hat{S} and A satisfy the Riccati-type equation

$$(1.5.1) \quad \hat{S}' + \hat{S}^2 + \{R(\cdot, \dot{\gamma}(t)) \dot{\gamma}(t)\}^h + 3AA^* = 0.$$

Next we show that

$$\max\{\text{eigenvalue } \hat{S}(\alpha)\} \leq \cot \alpha.$$

For $z \in H(\alpha)$, let $J \in \Lambda$ satisfy $J(\alpha) = z$. Then

$$(1.5.2) \quad \begin{aligned} g(\hat{S}(\alpha)z, z) &= g\left(\left((J^h)'\right)^h, z\right) \\ &= g\left((J^h)', z\right). \end{aligned}$$

If $J^v(t)$ denotes the component of $J(t)$ in $V(t)$, then for X tangent to H ,

$$g((J^v)', X)|_\alpha = -g(J^v, X')|_\alpha = 0,$$

since $J^v(\alpha) = 0$. In particular, $((J^v)')^h|_\alpha = 0$. Combined with Equation 1.5.2 this gives

$$\begin{aligned} g(\hat{S}(\alpha)z, z) &= g(J', z) \\ &= g(S(\alpha)z, z) \\ &\leq g(\cot(\alpha)z, z). \end{aligned}$$

Hence $\max\{\text{eigenvalue } \hat{S}(\alpha)\} \leq \cot \alpha$, as claimed.

Now set

$$\hat{R} \equiv \{R(\cdot, \dot{\gamma}(t)) \dot{\gamma}(t)\}^h + 3AA^*,$$

and for $x \in H(t)$,

$$\widehat{\sec}(x) \equiv g\left(\hat{R}\left(\frac{x}{|x|}\right), \frac{x}{|x|}\right).$$

Since AA^* is a nonnegative operator and $\sec(M) \geq 1$, $\widehat{\sec} \geq 1$. Substituting into Equation 1.5.1 we have

$$\hat{S}' + \hat{S}^2 + \hat{R} = 0.$$

Since $\widehat{\sec} \geq 1$ and $\max\{\text{eigenvalue } \hat{S}(\alpha)\} \leq \cot \alpha$, we apply the proof of Theorem H and conclude that $\hat{S} \equiv \cot(t) \cdot id$ and consequently $\widehat{\sec} \equiv 1$. Substituting back into $\hat{S}' + \hat{S}^2 + \hat{R} = 0$, we find

$$-\csc^2(t) \cdot id + \cot^2(t) \cdot id + \{R(\cdot, \dot{\gamma}(t)) \dot{\gamma}(t)\}^h + 3AA^* = 0$$

or

$$\{R(\cdot, \dot{\gamma}(t)) \dot{\gamma}(t)\}^h + 3AA^* = id.$$

Applying both sides to an $x \in H(t)$ and then taking the inner product with x gives

$$\begin{aligned} g(x, x) &= g(R(x, \dot{\gamma}(t)) \dot{\gamma}(t), x) + g(3AA^*(x), x) \\ &\geq g(x, x) + g(3AA^*(x), x). \end{aligned}$$

Thus

$$0 \geq g(3AA^*(x), x).$$

On the other hand, $g(AA^*(x), x) = g(A^*(x), A^*(x)) \geq 0$; so $AA^*x \equiv 0$, and $A \equiv 0$. It follows that $H(t)$ is parallel, and $\sec(x, \dot{\gamma}) \equiv 1$ for all $x \in H(t)$. So $H(t)$ is spanned by Jacobi fields of the form $\sin(t)E(t)$ where E is a parallel field tangent to H . \square

Jacobi Fields That Violate Our Hypotheses. The following example shows that the hypothesis in Theorem B about the Riccati operator being self-adjoint is necessary.

Example 1.6. Consider S^3 with the round metric. Define

$$J_1(t) \equiv \sin(t) \cdot E_1(t) + \cos(t) \cdot E_2(t)$$

and

$$J_2(t) \equiv \cos(t) \cdot E_1(t) - \sin(t) \cdot E_2(t)$$

where $E_1(t)$ and $E_2(t)$ are orthonormal parallel fields. Set $\Lambda \equiv \text{span}\{J_1, J_2\}$. Since $\{J_1(t), J_2(t)\}$ is orthonormal for all t , no Jacobi field $J \in \Lambda$ has a zero, so the conclusion of Theorem B does not hold for this family. On the other hand, neither do the hypotheses, since $S|_\Lambda$ is not self-adjoint. Indeed,

$$\begin{aligned} g(SJ_1, J_2) &= g(\cos(t) \cdot E_1(t) - \sin(t) \cdot E_2(t), \cos(t) \cdot E_1(t) - \sin(t) \cdot E_2(t)) \\ &= \cos^2(t) + \sin^2(t) \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} g(J_1, SJ_2) &= g(\sin(t) \cdot E_1(t) + \cos(t) \cdot E_2(t), -\sin(t) \cdot E_1(t) - \cos(t) \cdot E_2(t)) \\ &= -\sin^2(t) - \cos^2(t) \\ &= -1. \end{aligned}$$

The requirement that the maximum of the eigenvalues be bounded above is also necessary, and our hypothesis is the optimal one. This is demonstrated by the following example.

Example 1.7. Let $\gamma : (-\infty, \infty) \rightarrow S^n(1)$ be a geodesic in the unit sphere. Let $\{E_i\}_{i=1}^{n-1}$ be a basis of parallel fields along γ that are all normal to γ . For $\varepsilon \in (0, \frac{\pi}{6})$, set

$$J_i = \sin(t - \varepsilon) \cdot E_i(t),$$

and define $\Lambda \equiv \text{span}\{J_i\}$. Since $J_i(\varepsilon) = 0$ for all i , $S|_\Lambda$ is self-adjoint. Moreover, $S(t) = \cot(t - \varepsilon) id$. So for $\alpha = \frac{\pi}{2}$ we have

$$\max\{\text{eigenvalue } S\left(\frac{\pi}{2}\right)\} = \cot\left(\frac{\pi}{2} - \varepsilon\right) > 0 = \cot\left(\frac{\pi}{2}\right).$$

That is our family, Λ , does not satisfy our boundary hypothesis, $\max\{\text{eigenvalue } S(\alpha)\} \leq \cot \alpha$. On the other hand, each $J \in \Lambda \setminus \{0\}$ is nonvanishing on $[\alpha, \pi]$, so the conclusion does not hold either.

2. POSITIVE RICCI_k

Proof of Theorem C. Let

$$\mathcal{Z} \equiv \{J \in \Lambda \mid J(t) = 0 \text{ for some } t\}.$$

Suppose

$$\dim \mathcal{Z} \leq n - 1 - k.$$

Then

$$\dim \mathcal{Z}^\perp \geq (n - 1) - (n - 1 - k) = k$$

Apply Wilking's generalization of the horizontal curvature equation, Theorem 1.4, and the proof of Theorem D to conclude that Λ splits orthogonally into

$$\Lambda = \mathcal{Z} \oplus \mathcal{P}$$

where \mathcal{P} is a family of parallel Jacobi fields and has dimension $\geq k$.

In particular $\text{Ric}_k(\dot{\gamma})$ is not positive, so if $\text{Ric}_k(\dot{\gamma})$ is known to be positive, then it must be that

$$\dim \mathcal{Z} \geq n - k.$$

□

A nearly identical argument gives the proof of Theorem G, except the role of the proof of Theorem D is played by the proof of Theorem H. To apply Theorem H we need to establish the hypothesis on the upper bound on the initial values of the Riccati operator. This is achieved in precisely the same manner as in the proof of Theorem B.

Proof of Theorem F. Let $\gamma : (-\infty, \infty) \rightarrow E$ be a horizontal geodesic. Let Λ be the space of Jacobi fields along γ that arise from variations that leave $\pi^{-1}(\pi(\gamma(0)))$ orthogonally. Let

$$\mathcal{Z} \equiv \{J \in \Lambda \mid J(t) = 0 \text{ for some } t\}.$$

By Theorem C, $\dim \mathcal{Z} \geq n - k$.

On the other hand, according to Wilking (top of page 1305 of [Wilk1]), \mathcal{Z} is everywhere tangent to the dual leaf through $\gamma(0)$. Since \mathcal{Z} is everywhere orthogonal to γ and γ is everywhere tangent to the dual leaf, the dimension of the dual leaf is $\geq n - k + 1$. □

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